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# **IMPOSSIBILITY OF STRATEGY-PROOFNESS WITH COALITION FORMATION UNDER TRANSFERABLE UTILITY<sup>\*</sup>**

## **INTRODUCTION**

A society consists of a set of individuals  $I = \{1, \dots, N\}$  and a set alternatives  $A = \{a_1, \dots, a_M\}$ . Each individual  $i \in I$  has rational preference relation on  $A$ . A central issue here is that of aggregating individual preferences into a social preference ordering. Arrow showed that if the admissible domain of preferences is unrestricted and the number of alternatives is at least three, then the aggregate preference is Paretian and Independent of Irrelevant Alternatives if and only if there is an Arrovian dictator<sup>1</sup>. An important aspect of such aggregation programs is that the aggregator (or the planner) is not always aware of the true individual preferences and hence individuals may misrepresent their preferences to affect the social outcome. Such social choice problems may be interpreted as multi-criterial decision making where individual preferences are the criteria and the social planner is a decision maker. Gibbart [6] and Satterthwaite [12] (GS henceforth) showed that any mechanism that wishes to implement truth-telling (that is strategy-proofness) on part of the individuals and is unanimous must also be dictatorial in the sense of Arrow. Several attempts have been made to circumvent this dictatorial misfortune of such aggregation program<sup>2</sup>. In tandem, probabilistic social choice rules have also received significant attention, partly motivated by the desire to escape the dictatorial results generated in the deterministic framework<sup>3</sup>. Under such non-deterministic programs a new and significantly weaker version of dictator-

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<sup>1</sup> An Arrovian dictator is an individual  $h \in I$  such that for every individual preference profile of the society his individual preference is identical with the social preference.

<sup>2</sup> Among several possibility results are ones which allow the social preference to be less than fully rational or where individual preferences are single peaked.

<sup>3</sup> See for example [10; 7; 5; 1].

ship, called random dictatorship, is proposed (for example, see [10]). A probabilistic social choice rule is random-dictatorial if independent of the announcement of profile of preferences, there is a pre-determined probability for each individual with which that individuals most preferred alternative is ranked most preferred by the society as well. In a recent work by Dutta, Peters and Sen (DSP) [4] it is shown that if the planner wishes to implement a strategy-proof probabilistic social decision scheme (which is cardinal) and the number of alternatives is at least three, then the mechanism is unanimous if and only if is a random dictatorship.

In this note we test the GS and the DPS results in face of coalitional mis-reporting. A coalition is any non-empty subset of  $I$  who agree to jointly report their preference profile. Coalitional strategy-proofness has been studied in [3] and [8; 9]. All these papers defined coalitional strategy proofness as situation where for every preference profile of the society, for every non-empty coalition of individuals and for every joint deviation profile for such coalitions, there always exists some member of that coalition for whom the initial profile is not worse. With this approach, Dasgupta et. Al show that the social choice function is monotonic, the domain is rich and the preference domain has a product structure, then the social choice functions coalitionally strategy proof. Allowing for infinitely many individual, Mihara [8] show that any coalitionally strategy proof social choice function must depend only on the top most elements of each individual's preferences. In [9] a concrete example is provided for a coalitionally strategy proof non-dictatorial social choice in case of countably infinite societies. Such an existence was also shown in [11] but in a non-constructive manner.

The definition of coalitional strategy-proofness used in the papers cited above deals with non-transferable utility problems since it does not specify any notion of imputations or redistributions of worth's of coalitions amongst their numbers. In this note we instead concentrate on transferable utility scenarios. We consider two cases, one with a deterministic social choice function where GS is readily applicable and its probabilistic counterpart where DPS is the most natural extension of GS. We show that neither Arrowian nor random dictatorship are sufficient to guarantee coalitional strategy-proofness and particular that it one cannot guarantee to achieve truth-telling in environments where coalitions may be formed and utility is transferable. In the next section we define some terms and in section 3 we prove our main results.

## 1. DEFINITIONS

Let  $I = \{1, \dots, N\}$  and  $A = \{a_1, \dots, a_M\}$  be the set of individuals and alternatives as defined above. Each individual  $i \in I$  has a reference relation on  $A$  represented by a utility function  $u_i : A \rightarrow R$ . Let  $U$  denote the domain of all such utility functions normalized such that  $\max_A \{u_i\} = 1$  and  $\min_A \{u_i\} = 0$  for every  $i \in I$ .

A society is *cooperative* if there exists at least two individuals  $i, j \in I$  such that they reveal truthfully their individual utilities to each other and agree to announce jointly a utility pair  $(u_i, u_j) \in U^2$ .

A society is *minimally cooperative* if there are at most two individuals for whom the society becomes cooperative.

A society is *fully cooperative* if for any nonempty coalition  $S \subseteq I$ , each member  $i \in S$  reveals truthfully his utility to every  $j \in S \setminus \{i\}$  and the coalition  $S$  agrees to jointly announce a utility profile  $u_S = (u_i)_{i \in S} \in U^{|S|}$ .

A *cardinal social choice scheme* is a mapping  $\varphi : U^N \rightarrow O$  where  $O$  is the set of outcomes.  $\varphi$  is *deterministic* if  $O = A$  while it is *probabilistic* if  $O$  is the set  $L(A)$  of all possible lotteries over  $A$ <sup>4</sup>.

Let  $V_{u_i}$  be a *value operator* under the utility function  $u_i$ . That is  $V_{u_i} : O \rightarrow R$ . We consider only two cases for  $V_{u_i}$ : (i) if  $O = A$  then  $V_{u_i}(a_t) = u_i(a_t), a_t \in A$ , while (ii) if  $O = L(A)$  then  $V_{u_i}$  is the mathematical expectation under  $u_i$  given a lottery  $\lambda$  in  $L(A)$ . That is:

$$V_{u_i}(\lambda) = \sum_{t=1}^M \lambda_t u_i(a_t)$$

where  $\lambda_t$  is the probability of alternative  $a_t \in A$  under the lottery  $\lambda$  in  $L(A)$ .

Given a utility profile  $u \in U^N$  and any coalition  $S \subseteq I$ , the profile  $w = (u_S, u_{-S})$  is:

$$w_j = \begin{cases} u_j & \text{if } j \notin S \\ u_j & \text{if } j \in S. \end{cases}$$

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<sup>4</sup> There are other versions of deterministic schemes, for example, set-valued outcomes. We restrict attention to singleton-set outcomes here.

A social choice scheme  $\varphi$  is Individually *Strategy-proof* if for all  $i \in I$ , for all  $u \in U^N$  and for all  $u'_i \in U$  we have  $V_{u_i}(\varphi(u)) \geq V_{u'_i}(\varphi(u'_i, u_{-i}))$ .

We now come to our transferable-utility notion of coalitional strategy proofness.

A social choice scheme  $\varphi$  is *coalitionally manipulable* by some non-empty coalition  $S \subseteq I$  at the utility profile  $u \in U^N$  via a joint announcement  $u'_S \in U^{|S|}$  if  $\exists (x_i)_{i \in S}, x_i \in R$  such that:

$$(i) \quad \sum_{i \in S} x_i \leq \sum_{i \in S} V_{u_i}(\varphi(u'_S, u_{-S})),$$

$$(ii) \quad x_i \geq V_{u_i}(\varphi(u)) \quad \text{for every } i \in S \quad \text{with strict inequality for at least some } i \in S.$$

Consequently, a social choice scheme  $\varphi$  is *coalitionally strategy proof* if it is not coalitionally manipulable at any utility profile by any coalition.

### Remark 1

If  $\varphi$  is Coalitionally Strategy-proof then must it be Individually Strategy-proof.

We now state clearly the GS and the DPS theorems in our setting.

### GS Theorem

Let  $\varphi: U \rightarrow A$  be a deterministic social choice scheme with  $|A| \geq 3$ . Then  $\varphi$  is Individually Strategy-proof if and only if it is dictatorial *a la Arrow*.

Consider any probabilistic cardinal social choice scheme  $\varphi: U \rightarrow L(A)$ . Then  $\varphi$  is a *random dictatorship* if independent of the profile  $u \in U^N$ , there exists non-negative real numbers  $\delta_i$  with  $\sum_{i \in I} \delta_i = 1$  such that:

$$\varphi^t(u) = \sum_{i \in \{j \in I \mid u_j(a_i) = 1\}} \delta_i$$

where  $\varphi^t(u)$  is the probability that  $\varphi$  attaches to the alternative  $a_t \in A$  under the profile  $u \in U^N$ .

### DPS Theorem

Let  $\varphi : U \rightarrow L(A)$  be a probabilistic social choice scheme with  $|A| \geq 3$ . Then  $\varphi$  is Unanimous and Individually Strategy-proof if and only if it is a random dictatorship.

## 2. MAIN RESULT

### Theorem 1

Let  $\varphi : U \rightarrow O$  be a cardinal social choice scheme with  $|A| \geq 3$ . Then  $\varphi$  cannot be Coalitionally Strategy-proof even in a Minimally Cooperative Society.

### Proof

By the GS and the DPS theorems, it is sufficient to consider dictatorial schemes. The proof is by example. Since  $U$  is an unrestricted domain, consider the following preference profile:

$$\begin{bmatrix} 1 & 2 & \dots & N \\ a & b & . & . \\ b & . & . & . \\ . & a & . & . \end{bmatrix}$$

Here for each individual 1, 2, ..., and so on, the top element above is the best etc. Without any loss of generality, assume that individual 1 receives a utility equal to  $\alpha$  from alternative b. Now, consider any dictatorial scheme such that  $\delta_i \in [0,1]$  and  $\sum_{i \in N} \delta_i = 1$ . Note, that this include Arrovian and Random dictatorships. Suppose the coalition {1,2} is formed.. Truth telling yields a total payoff of:

$$\delta_1 + \delta_2(1 + \alpha) + B$$

where  $B = \sum_{i=3}^N \delta_i [u_1(\tau(\succ_i)) + u_2(\tau(\succ_i))]$  and  $\tau(\succ_i)$  is the top element of individual  $i$ 's preference  $\succ_i$ . Consider the following mis-representation by the coalition {1,2}:

$$\begin{bmatrix} 1 & 2 & \dots & N \\ b & b & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & a & \cdot & \cdot \end{bmatrix}$$

Then, total payoff for the coalition  $\{1,2\}$  from this misrepresentation, given the previous  $\delta_i$  is:

$$\delta_1(\alpha+1) + \delta_2(\alpha+1) + B$$

Clearly, for any  $\alpha > 0$  we have:

$$\delta_1(\alpha+1) + \delta_2(\alpha+1) + B > \delta_1 + \delta_2(1+\alpha) + B$$

which completes the proof.

The theorem shows that whenever utility is expressed in terms of money, strategy-proofness is an impossible virtue.

## CONCLUSIONS

The transferable utility may be interpreted as utility measured in terms of money, and money is transferable. Hence any society where there is money, one cannot guarantee truth-telling in case of coalition formation for problems where utility is equivalent to money. However, there are problems where utility is not transferable, for example, happiness, fear etc. In such cases, our results does not apply.

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